Stability conditions on derived categories

Set $X = \text{tot}(\mathcal{O}_{\mathbb{P}^2}(-3))$ - a non-compact Calabi-Yau threefold

Consider $\mathcal{D}^b(\text{Coh}(X))$, the bounded derived category of coherent sheaves on $X$.

Let $\mathcal{D} \subseteq \mathcal{D}^b(\text{Coh}(X))$ be a full subcategory consisting of objects supported on $\mathbb{P}^2 \subset X$.

**Aim:** Understand mathematically $\Pi$-stability and the stringy Kähler moduli space in this example.

First observation: $X$ is a crepant resolution of an orbifold

\[ \mathbb{C}^3 \xrightarrow{(2,1,3)} X \]

(2,1,3) blow-up

of the singularity $\mathbb{C}^3/(\mathbb{Z}/3)$ into $\mathbb{P}^2$.

The action is given by

\[ (2,1,3) \to \quad (w^2, w^2, w^2), \quad w = e^{2\pi i/3} \]
The McKay correspondence gives an equivalence

\[ D^b \text{Coh} \frac{\mathbb{C}^3}{\mathbb{Z}/3} \stackrel{\sim}{\rightarrow} D^b \text{Coh} X \]

equivariant coherent sheaves

It is easy to see that \( \text{Coh} \frac{\mathbb{C}^3}{\mathbb{Z}/3} \cong \text{Mod}(A) \) for the non-commutative ring \( A \) given by the McKay quiver.

This is the quiver

\[ \begin{array}{c}
  \bullet & \overset{x_1}{\rightarrow} & \bullet \\
  \downarrow z_3 & & \downarrow \ \ \ \\
  \bullet & \overset{x_1}{\rightarrow} & \bullet \\
\end{array} \]

with relations

\[ x_i y_j = y_i x_j \]
\[ x_i z_j = z_i x_j \]
\[ y_i z_j = z_i y_j \]

This can be easily gotten by noting that an equivariant coherent sheaf on \( \mathbb{C}^3 \) is a module for the polynomial algebra in 3 variables and the group \( \mathbb{Z}/3 \).
These relations can be packaged in a superpotential

$$\Phi = \sum_{i,j,k} \epsilon_{ijk} x_i y_j z_k$$

The relations are just $d\Phi = 0$.

These equivalences restrict to give an equivalence

$$\Phi: \text{Db } \text{Mod}_0(A) \sim \textbf{D}$$

where $\text{Mod}_0(A) \subset \text{Mod}(A)$ is the full subcategory of nilpotent $A$-modules.

Note: $A$ is graded by path length

$$A = \bigoplus_{n \geq 0} A_n$$

$u$ = path length of monomials

A module $M \in \text{Mod}(A)$ is nilpotent if and only if

$$A_n M = 0 \quad \forall n > 0$$

The category $\text{Mod}_0(A)$ is finite length with three simple objects. Let $S_0, S_1, S_2$ be the images of these simple objects under $\Phi$. 
The objects $S_i$ are spherical, i.e. they have the cohomology of a 3-sphere:

$$\text{Hom}_D(S_i, S_j) = \begin{cases} \mathbb{C} & i = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

In particular, $S_i$ give rise to Seidel-Thomas twists $T_{S_i} \in \text{Aut}(D)$.

By definition $T_{S_i}$ sends any object $E \in D$ to the object $T_{S_i}(E) \in D$ given as a cone:

$$\text{Hom}(S_i, E) \otimes S_i \rightarrow E$$

In this case, the stringy Kähler moduli is a 2-sphere with 3 special points:
Here the decorations under the points are the local monodromies thought of as elements in Aut(\(\Omega\)).

Note that the orbifold point of the Kähler moduli is a cusp singularity (with stabilizer \(\mathbb{Z}/3\)). So the 3-sheeted cover above is ramified only at LV point and is unramified everywhere else.

One can show that

\[
\langle T_{\bar{g}_0}, T_{\bar{g}_1}, T_{\bar{g}_2} \rangle = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \subset Aut(\Omega)
\]

In fact \(\langle T_{\bar{g}_i} \rangle\) acts freely on \(K(\Omega) \cong \mathbb{Z}^3\).
The mirror of $X$ is

$$X = (uv = x+y + \frac{1}{xy}) = \mathbb{C}^2 \times (\mathbb{C}^\times)^2$$

$$X \rightarrow (\mathbb{C}^\times)^2$$

$$\downarrow \quad \downarrow \quad \downarrow x+y + \frac{1}{xy} \quad \text{mirror of}$$

$$\mathbb{C}^2 \rightarrow uv \quad \text{p}^2$$

Notice that $f$ has 3+1 critical values

![Diagram]

In fact there are many algebras $\mathcal{B}$ with the property that

$$D^b \text{ Mod}_0 (\mathcal{B}) \rightarrow \mathcal{D}$$

The family of such $\mathcal{B}$'s has a complicated combinatorial symmetry governed by the group

$$G = \langle \tau_0, \tau_1, \tau_2 \mid \tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j, \forall i, j \rangle$$
Note that \( G \subset T \), Config \( 3 (C^2) \) and
\( T_0, T_1, T_2 \) correspond to the loops exchanging 3 - pts as follows:

\[ \text{Diagram} \]

Theorem: For each \( g \in G \), there is an algebra \( A(g) \) defined by a quiver (with relations) of the form

\[ \begin{align*}
\begin{array}{c}
\cdot \\
\downarrow \quad \downarrow \\
\cdot \\
\end{array} \\
\begin{array}{c}
a \\
b \\
c \\
\end{array}
\end{align*} \]

\[ a^2 + b^2 + c^2 = abc \]

The relations for each such quiver are given by a superpotential which is uniquely determined by the triple \((a, b, c)\).

For each \( g \in G \), there is an equivalence
\[ D_g : \text{D}^b \text{Mod}_g A(g) \to D \]

and these equivalences are all different in the sense that abelian subcategories
$\mathcal{H}(g) \subset D$ given by $\mathcal{H}(g) = \Phi_g \text{ Mod}_0(A(g))$

are all different abelian subcategories in $D$.

In fact we can order the simples for each $A(g)$ and then:

$$0 \rightarrow \cdots \rightarrow \mathcal{S}_1 \rightarrow \mathcal{T}_1 \rightarrow \mathcal{S}_0 \rightarrow (w')^* \rightarrow \mathcal{T}_0^*(\mathcal{S}_2)$$

Superpotential $U \otimes V \otimes W \rightarrow C$ (for $A(g)$)

given by

$$\text{Ext}(\mathcal{S}_0, \mathcal{S}_1) \otimes \text{Ext}^1(\mathcal{S}_1, \mathcal{S}_2) \otimes \text{Ext}^1(\mathcal{S}_2, \mathcal{S}_0) \rightarrow \text{Ext}^3(\mathcal{S}_0, \mathcal{S}_0) \rightarrow C$$

Superpotential $U^* \otimes V^* \otimes W^* \rightarrow C$ (for $A(\bar{\tau}_g)$)

where

$$0 \rightarrow (w')^* \rightarrow U \otimes V \rightarrow W^* \rightarrow 0$$

Note: The operation $A(g) \rightarrow A(\bar{\tau}_g)$ changes the number of arrows between simples:
The operation of passing between $A(g)$ and $A(\tilde{v}_{i+1} g)$ is called a tilt ($= \text{Seiberg duality}$):

$$\text{Mod}_0(A)[\mathbb{Z}] \quad \text{Mod}_0(A) \quad \text{Mod}_0(A)[\mathbb{Z}]$$

\[ D = \mathbb{Z} \cdot \text{Mod}_0(A) \]

The transition $\text{Mod}_0(A) \rightarrow \text{Mod}_0(B)$ is the tilt. This was invented by Breen - Butler in the late 70s.

Note: The tilts of a 3d non-compact CY category form a very regular pattern. This is not the case in general.
There is a group homomorphism

\[ \phi : G \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \]

such that

\[ \phi(g_1) = \phi(g_2) \iff \text{Mod}_0(A(g_1)) \cong \text{Mod}_0(A(g_2)) \]

\[ \Rightarrow \]

\[ \begin{array}{c}
\text{Db Mod}_0(A(g_1)) \\
\downarrow \Phi_{g_1} \\
\text{Db Mod}_0(A(g_2)) \end{array} \xrightarrow{\Phi_{g_2}} \begin{array}{c}
\text{D} \\
\downarrow \Psi \\
\text{D} \end{array} \]

\[ \Psi \in \langle T_{g_0}, T_{g_1}, T_{g_2} \rangle \subset \text{Aut}(\text{D}) \]

Modding out by \( \langle T_{g_0}, T_{g_1}, T_{g_2} \rangle \) gives the Cayley graph of

\( (\mathbb{Z}_2)^3 \times (\mathbb{Z}_2)^3 \times (\mathbb{Z}_2)^3 \)
Last time we considered

\[ D = D^b \text{coh} \left( \text{op}(\mathcal{A}) \right) \]

full subcategory of objects supported on the zero section and considered different abelian subcategories in \( D \) of the form \( \text{Mod}_0(\mathcal{A}) \) and such that

\[ D = D^b \text{Mod}_0(\mathcal{A}). \]

Now we will look at these abelian categories more systematically.

Let \( D \) be a triangulated category

**Def**: A **stability condition** on \( D \) consists of a group homomorphism \( \zeta : K(D) \to \mathbb{C} \) (called the central charge) and full subcategories \( \mathcal{P}(\phi) \subset D \) for all \( \phi \in \mathbb{R} \) (the semi-stables of phase \( \phi \)) such that

1. If \( E \in \mathcal{P}(\phi)[1] \), then \( \zeta(E) \in \mathbb{R} > 0 \).
2. \[ \mathcal{P}(\phi)[1] = \mathcal{P}(\phi+1) \]
3. If \( \phi_1 > \phi_2 \) and \( A_i \in \mathcal{P}(\phi_i) \), then \( \text{Hom}_D(A_i, A_2) = 0 \).
4. If \( 0 \neq E \in D \), \( \phi_2 > \phi_2 > \ldots > \phi_n \) are triangles.
0 : E_0 \to E_1 \to E_2 \to \cdots \to E_n \to E_n = E

\begin{array}{c}
\mathcal{F} \setminus \mathcal{D}
\end{array}

A_1 \quad A_n

with \ A_i \in \mathcal{D}(\phi_i)

(degree chain filtration).

Now let \( \text{Stab}(D) \) to be the set of all stability conditions on \( D \) that are locally finite.

There is a natural topology on the set \( \text{Stab}(D) \) so that we have

\textbf{Theorem:} For each connected component

\( \Xi \in \text{Stab}(D) \) there exists a linear subspace

\( V(\Xi) \subset \text{Hom}_\mathbb{R}(K(D), \mathbb{C}) \)

with a well-defined linear topology such that the forgetful map

\[ \Xi : \Xi \to \text{Hom}_\mathbb{R}(K(D), \mathbb{C}) \]

\[ (\Xi, \phi) \mapsto \Xi \]

is a local homeomorphism onto an open
This implies that $\text{Stab}(\Sigma)$ is a complex manifold (possibly infinite dimensional).

We can also look at finite dimensional slices of this space.

Suppose now it is a finite length abelian category with simples $S_0, S_1, S_2$.

Suppose $z : K(A) \to E$ is a group homomorphism such that

$$z(E) \in \mathcal{E} = \left\{ \varepsilon \in E \mid \right. \begin{array}{c} -i \phi(E) \leq 1 \\ 0 < \phi(E) \leq 1 \end{array} \left. \right\}$$

for all $E \in \mathcal{E}$.

Note: It is enough to check this for $S_i$. We will represent the corresponding data by

$$z(S_i) \to z(S_0)$$
Define
\[
\mathcal{P}(\phi) = \left\{ E \in \mathcal{A} \mid 0 \leq \phi \leq 1 \text{ and } 0 \neq A \in E \implies \mathcal{P}(A) \leq \mathcal{P}(E) \right\} \\
\mathcal{P}(E) = \phi
\]
for \( 0 < \phi < 1 \).

This data can be extended to a stability condition on \( D = D^b(\mathcal{A}) \).

Indeed:
- \( K(D) = K(\mathcal{A}) \)
- (b) gives \( \mathcal{P}(\phi) \) for all \( \phi \)
- the filtrations in (d) come from Harder-Narasimhan filtrations in it
- truncation functors for the standard + -structure on \( D^b(\mathcal{A}) \)

Remark: Conversely, given \((E, \mathcal{P})\) on \( D \), then
\[
\mathcal{P}(0,1) = \{ E \in D \mid 1 > \phi, > \phi, > \phi > 0 \}
\]
is an abelian category (in fact, it is the heart of a bounded t-structure possibly unfaithful one). Then it is \( \mathcal{P}(0,1) \) reverses the above process in some sense.
Take $D = D^b \text{Coh}(\text{Ope}(-3))$ as before.

Recall that we had equivalences

$$ \phi_g : D^b \text{Mod}_A(g) \xrightarrow{\sim} D $$

indexed by

$$ g \in G = \langle \tau_0, \tau_1, \tau_2 \mid \text{braid relations} \rangle $$

**Theorem:** There is a subset $\text{Stab}^0(D) \subset \text{Stab}(D)$ which as a set is

$$ \text{Stab}^0(D) = \bigsqcup_{g \in G} U(g) $$

with each $U(g) = \mathcal{L} \times \mathcal{L} \times \mathcal{L}$. Moreover

$$ U(g_1) \cap U(g_2) $$

is codimension one in $\text{Stab}^0(D)$ iff $g_1 g_2^{-1} = \tau_i \tau_i^{-1}$

For instance,

$$ z(\$0) \xrightarrow{z(\$1)} z(\$2) \in U(g) $$

$$ z(\$0) \xrightarrow{z(\$1)} z(\$2) \notin U(g) \setminus U(g) $$

$$ z(\$2) \xrightarrow{z(\$1)} z(\$0) \in U(g_1) $$

$$ z(\$2) \xrightarrow{z(\$1)} z(\$0) \notin U(g_1) $$
The previous theorem implies that $z(s_0), z(s_1)$ and $z(s_2)$ are local coordinates on $\text{stab}_0(D)$.

However, these coordinates do not reflect the $G$-action. So we want to find new coordinates in which the $G$-action is visible.

Dubrovin showed that the quantum cohomology of $\mathbb{P}^2$ extends to give a Frobenius structure on a dense open subset

$$M = \text{Conf}_3 \left( \mathbb{C} \right) = \{ (u_0, u_1, u_2) \in \mathbb{C}^3 | u_i \neq u_j \}_{i \neq j}$$

where $u_i$'s are the canonical coordinates

$$\frac{2}{\partial u_i} \frac{2}{\partial u_j} = \delta_{ij} \frac{2}{\partial u_i} \frac{2}{\partial u_i}$$

The discriminant $\Delta = U_i (u_i = 0) \subset M$ for $i = 0, 1, 2$.

and Dubrovin used the Frobenius structure to define a twisted period map

$$\Delta \setminus M \rightarrow \mathbb{C}^3$$
On the subset \( M_k (X) \)

\[ \mathbb{P}^1 - 1, 0, 1, \ast y \leftrightarrow M \backslash \Delta \]

\[ z \rightarrow \frac{1}{z} + 2 \frac{z}{y} \]

the period map satisfies the Picard-Fuchs equation

\[
\left[ \theta_2^3 - z (\theta_2 + \frac{1}{3}) (\theta_2 + \frac{2}{3}) \theta_2 \right] W = 0
\]

where \( \theta_2 = \frac{d}{dz} \)

**Note:** \( M_k (X) \) is the kähler moduli space on \( X = \mathbb{C}^2 / \mathbb{Z}^2 \)

**Guess:** \( \text{Stab} (\mathcal{D}) \cong M \backslash \Delta \)

\[ (\gamma, z) \begin{array}{c} z \rightarrow \sqrt{z} \\ \mathbb{C}^3 \end{array} \]

\[ (\frac{\theta_0}{2}, 2 \frac{\theta_1}{\gamma}, 2 \frac{\theta_2}{\gamma}) \]

Here \( \frac{\theta_0}{2}, \frac{\theta_1}{\gamma}, \frac{\theta_2}{\gamma} \) are objects in \( \mathcal{D} \)

whose \( K \)-classes are a basis on \( K(\mathcal{D}) \).
Conclusions:

(a) $M_k(X)$ sits inside a bigger $3D$ space ($= M - D$) which is analytically continued in big quantum cohomology.

(b) $\text{Stab}(D)$ seems to have a Frobenius structure. Can we get this structure from quiver representations?