"Quivers, exceptional collections
and projective toric manifolds."

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(joint work with GREGORY SMITH)
§1 Motivation.

$X$ smooth projective variety / $\mathbb{C}$

$L_0, L_1, \ldots, L_r \in \text{Pic}(X)$

$T = \bigoplus_{i=0}^r L_i$, \quad $A = \text{End}(T)$.

$T$ is a [tilting bundle] if $(L_0, L_1, \ldots, L_r)$ form a full strong exceptional collection on $X$

$$\Rightarrow \mathbb{D}^b(\text{Coh}(X)) \xrightarrow{\text{RHom}(T,-)} \mathbb{D}^b(A\text{-mod})$$

are mutually inverse equivalences of triangulated categories.

[Observation [Bondal]]

The correct tool to study existence and properties of $T$ is the theory of [quivers].
Example 1

\[ X = \mathbb{P}^2 \]

Beilinson resolution \( \Rightarrow T = \bigoplus_{i=0}^\infty \mathcal{O}(i) \) is tilting

\[ A^\op = \bigoplus_{ij=0} \text{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \]

Q: \[ \begin{array}{c}
0 \xrightarrow{x_i} 1 \xrightarrow{y_i} ?
\end{array} \]

R: \[ x_i y_j = x_j y_i \]

\[ A^\op = CQ/\langle R \rangle. \]

Example 2

\begin{align*}
\text{(Kapranov-Vasserman)} \\
\text{(Bridgeland-King-Reid)} \\
\text{Bridgeland}
\end{align*}

\[ X \longrightarrow \mathbb{C}^n/G \] crepant resolution \( G=SL(n) \)

\( n=2,3. \)

\[ D^b(\text{Coh}(X)) \cong D^b(A\text{-mod}) \]

for

\[ A = C[x_1, \ldots, x_n] \times G \]

key: construct \( X \) as fine moduli space of representations of the McKay quiver w/ relations of the McKay quiver.
Conjecture [King]
X smooth proj. toric variety / \mathbb{C}. Then X admits a tilting bundle \( T = \bigoplus \mathcal{L}_i \).
(evidence - smooth Fano toric surfaces, \( \mathbb{P}^n, \mathbb{F}_n \))

**Definition** X smooth projective toric variety / \mathbb{C}
A collection \( \{ \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_r \} \) of effective line bundles form a **geometric collection** if
\[ \exists \ \theta_1, \ldots, \theta_r > 0 \]
(+ certain \( \theta_r > 0 \); to be specified) such that
1) \[ L = \mathcal{L}_1^{\theta_1} \otimes \cdots \otimes \mathcal{L}_r^{\theta_r} \text{ is ample (and gives a projectively normal embedding of } X) \]
2) \[ H^0(\mathcal{L}_1)^{\theta_1} \otimes \cdots \otimes H^0(\mathcal{L}_r)^{\theta_r} \rightarrow H^0(L) \]
is surjective - "regularity".

**Thm** [C-Smith, surely known to Bondal!]
Every smooth Fano 3-fold admits a tilting bundle whose summands form geometric collection.
§2 The Bondal quiver with relations

\{ L_0 = \mathcal{O}_X, L_1, \ldots, L_r \} effective line bundles on \( X \).

The Bondal quiver \( Q = Q(\{L_i\}) \) has

\[ Q_0 = \{ 0, 1, \ldots, r \} \]

\[ Q_1 = \left\{ i \rightarrow j : \begin{array}{c}
\text{torus-inv. } s \in H^0(L_j \otimes L_i) \\
\text{doesn't factor through some } L_k
\end{array} \right\} \]

Label arrow with effective divisor \( \text{div}(s) \).

To construct the relations \( R : \)

for \( i, j \in Q_0 \), any two paths from \( i \) to \( j \) labelled with same divisor give a relation.

Lemma

The path algebra of \( (Q, R) \) is

\[ \frac{\mathbb{C}Q \langle R \rangle}{\mathbb{C}Q \langle R \rangle} \cong \text{End} \left( \bigoplus_{i \in Q_0} L_i \right)^\circ = A^\circ \]
**Examples on** $X = \mathcal{F}_i = dP_i$

Let $L = \mathcal{O}_x(1,1)$ denote bundle giving embedding

$X \hookrightarrow \mathbb{P}^4$

**Example** - :

![Diagram 1](image)

No relations

**Example 0** :

![Diagram 2](image)

Three relations
(tilting bundle)

**Example +** :

![Diagram 3](image)

Many relations!
Encode Bondal quiver with labels in matrix:

\[ E : \mathbb{Z}^{Q_1} \longrightarrow \mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{\Delta_1} \quad \text{\text{\text{\text{\text{\text{\text{\text{$\Delta$ fan of $X$}}}}}}}} \]

\[ e_{(i \rightarrow j)} \longmapsto (e_j - e_i, \operatorname{div}(s)) \]

As matrix

\[ E = \begin{pmatrix} D \\ F \end{pmatrix} \quad \text{incidence matrix of $Q$} \]

\[ \text{columns record divisor labels} \]

**Example**

\[
E = \begin{pmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]
§3 Geometric quotient constructions of $X$

Representation of $Q$ of dim. vector $(1,1,..,1)$ is a point $w \in \bigoplus \operatorname{Hom}(W_i, W_j) \quad W_i \cong C \cong C^{q_1}$

Relations $R$ cut out subscheme $\mathbb{Z} \subset C^{q_1}$

Change basis via $T_D := (\mathbb{C}^*)^{q_0}/\mathbb{C}^*$ w/ character

Lattice $\mathbb{Z}D := $ image of $D: \mathbb{Z}^{q_1} \to \mathbb{Z}^{q_0}$

[King]: Given $\Theta \in \mathbb{Z}D$, obtain moduli space

$M_\Theta = \mathbb{Z}/\Theta T_D$ of $\Theta$-stable representations of $(Q, R)$

Remark: Given $\{ L_0, L_1,.., L_r \}$ and $x \in X$, the space $\bigoplus_{i,j} \operatorname{Hom}((L_i)_x, (L_j)_x)$ is rep of $(Q, R)$, so obtain

$X \longrightarrow \begin{cases} \text{moduli stack of } \\ \text{reps of } (Q, R) \end{cases}$

and ask:

does $\exists \Theta$ s.t. $X \longrightarrow M_\Theta$ ?

is this map isomorphism? on some component?
Warning: In general, $Z$ is ugly!

Even for

the scheme $Z$ has two irreducible components:

\[ I_Z = \langle z_1 z_5 - z_3 z_6, z_2 z_5 - z_6 z_7, z_1 z_4 z_7 - z_2 z_4 z_6 \rangle \]

Enter the hero!

Consider the toric variety $V \subseteq \mathbb{C}^Q$ defined by the toric ideal

\[ I_V = \langle \bar{z}^u - \bar{z}^v \in \mathbb{C}[\mathbb{C}^Q] : u - v \in \ker(E) \rangle \]

This is a $T_D$-invariant affine subvariety of the scheme $Z$, hence

\[ V_{/\theta T_D} \subseteq Z_{/\theta T_D} = M_{/\theta} \]

Remark: Unknown (hard problem!) whether $V$ is normal toric variety.
Reminder: \( \{L_0, L_1, \ldots, L_r\} \) effective line bundles form a geometric collection if \( \exists \Theta_1, \ldots, \Theta_r \geq 0 \) with \( \Theta_i > 0 \) for each sink \( i \in Q_0 \), such that

1. \( L = L_1^{\Theta_1} \otimes \cdots \otimes L_r^{\Theta_r} \) is ample (+ proj normal)
2. regularity: \( H^0(L_1)^{\Theta_1} \otimes \cdots \otimes H^0(L_r)^{\Theta_r} \rightarrow H^0(L) \).

Note: Given such \( \Theta_1, \ldots, \Theta_r \), set \( \Theta_0 = -\sum_{i=1}^r \Theta_i \),

so \( \Theta = (\Theta_0, \Theta_1, \ldots, \Theta_r) \in \mathbb{Z}^D \).

Example: \( \Theta = (-1, 1) \)

\( \Theta = (-1, 0, 0, 1) \)

\( \Theta = (-1, 0, 0, 0, 0, 1) \).

Theorem 2

If \( \{L_0, L_1, \ldots, L_r\} \) is a geometric callin then

\[ X = \bigvee \big/_{\Theta} T_D \]

is a geometric quotient, for \( \Theta \in \mathbb{Z} D \) given (in definition of geom collection).
Sketch proof:

1. Construct commutative diagram w/ exact rows

\[\begin{array}{c}
0 \rightarrow M \rightarrow \mathbb{Z}E \rightarrow \mathbb{Z}D \rightarrow 0 \\
\| \quad \| \\
0 \rightarrow M \rightarrow \mathbb{Z}A^1 \rightarrow \text{Pic}(X) \rightarrow 0
\end{array}\]

where \(\lambda(\theta_0, \theta_1, \ldots, \theta_r) = L_{\theta_0} \otimes L_{\theta_1} \otimes \ldots \otimes L_{\theta_r}\)

is ample for \(\Theta = (\theta_0, \ldots, \theta_r)\) given in geometric cells.

2. Establish graded-ring isomorphism

\[
\bigoplus_{k \geq 0} C[V]_{k\Theta} \cong \bigoplus_{k \geq 0} C[C^{Z_i}]_{L^k} \\
\cong \bigoplus_{k \geq 0} H^0(X, L^k)
\]

hence \(\mathcal{V}/\Theta T_D \cong X\) (cat. quotient)

3. For particular choice of \(\Theta\), every \(\Theta\)-semistable pt of \(Z\) is \(\Theta\)-stable, so

\[\Rightarrow \Theta\text{ generic, hence geometric quotient.}\]
Established same diagram for McKay quiver case

→ closed strings vs open strings!

Question: When is $X \cong M_\theta$?

... boils down to ...

Question: When is the inclusion

$$V/\theta T_D \hookrightarrow \mathbb{Z}/\theta T_D$$

an isomorphism (on some component of $\mathbb{Z}/\theta T_D$)

Warning

$\mathcal{O}_X \otimes \mathcal{O}_L \otimes X = \mathbb{F}_1$

No relations, but $\ker(E) \neq 0$, so

$V \not\cong \mathbb{C} = \mathbb{C}^5$

have different dimensions!
§4 Fine moduli construction.

Given any effective \( \{ L_0, L_1, \ldots, L_r \} \), Butler-King constructed complex

\[
\cdots \rightarrow \bigoplus_{i \in \mathbb{R}} L_j \boxtimes L_i^{-1} \xrightarrow{d_2} \bigoplus_{i \rightarrow j \in Q_1} L_j \boxtimes L_i^{-1} \rightarrow
\]

\[
\cdots \rightarrow \bigoplus_{i \in Q_0} L_i \boxtimes L_i^{-1} \xrightarrow{d_0} \mathcal{O}_X \rightarrow 0
\]

**Theorem 3**

- If effective collection \( \{ L_0, \ldots, L_r \} \) satisfies
  
  - \( \text{im}(d_2) = \ker(d_1) \)
  
  - \( Q \) has unique sink

  then \( V \in Z \) is the unique irreducible comp.

lying in no coordinate hyperplane of \( C^Q \).

- If in addition \( \{ L_0, \ldots, L_r \} \) is geometric, then

  \[
  X = V / G \quad \text{is "coherent component" of} \quad M_\theta = Z / G \quad T_0
  \]

  and inherits tautological line bundles from \( M_\theta \)

  \[\Rightarrow \quad X \text{ is (comp. of) fine moduli } M_\theta \]
Closing remarks:

1. In McKay quiver case \([CMT]\), 3 example with \(M_0\) not irreducible \(- G \subset \text{GL}(3)\) order 14 - so \(X \neq M_0\).

Hope to investigate Bondal quiver case (+ Proudfoot).

2. If \(\oplus\) Li tilting with geom collection \(\{L_0, \ldots, L_r\}\) on Fano surface or 3-fold toric, then

\[
X = M_0
\]

(other components of \(X\) are \(\Theta\)-unstable).

Expect to

3. Can extend to \(X\) simplicial / relatively proj.

in many cases.

4. Beautiful relationship with Mori- theory (birational geometry) appears in examples.