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A vertex formalism for degenerate torus actions

joint with Bogdan Florea and Natalia Saulina

Gromov-Witten theory

Count maps

Moduli space

\[ \overline{M}_{0,0}(X, \beta) \]

vir dim \[ \overline{M}_{0,0}(X, \beta) = 0 \]

Vir cycle \[ \left[ \overline{M}_{0,0}(X, \beta) \right] \in A_0 \]

\[ \overline{M}_{0,0}(X, \beta) = \{ f : \Sigma \to X \mid f_* [\Sigma] = \beta \} \]

\[ C_{\alpha \beta} = \int \left[ \overline{M}_{0,0}(X, \beta) \right] \]

\[ \sum_{\alpha \beta} C_{\alpha \beta} g^\alpha \omega^{-2} g^\beta \]
Atiyah-Bott localization

\[ \Gamma_x X \to X \] \(-\) torus action
\[ \lambda \in H^+(X) \] \(\text{source}\)

\[ \int_X \sum_{\text{fixed loci}} \frac{d\lambda E}{e^{-\lambda}(N^\mathbb{C})} \]

Equivalent Euler characteristic.

On stacks there is a version of this formula due to Gromov-Witten

\[ \int \frac{1}{[M]_\tau} \sum \frac{1}{[\mathfrak{g}] \ e^{-\lambda}(N^{\mathfrak{g}})_\mathfrak{g}^\mathbb{C}} \]

These formulas are the main computational tool.

Known facts:

\(a\) If \(X = \mathbb{Z}\)

\[ \text{hypersurface \ to \ toric \ variety} \]
We can use localization on $\mathbb{Z}$ to get a mirror theorem for $h=0$. Nothing is known for $h>0$.

(b) $X$ is a toric CY$_3$ (necessarily quasi-projective).

Here the $A$-model can be solved completely to all genera. This is done by a physicists version of localization called the topological vertex.

(This was invented by Aganagic - Kleb - Marino - Vafa
M. Li - K. Liu - J. Li)

Suppose $X$ has a torus action

$T \times X \rightarrow X$

which has only isolated fixed points.

These are necessarily joined by $T$-equivariant $\mathbb{P}^1$'s and the computation localizes onto these curves.
Here $E \sim M_{0, \infty}(pt)$

Deligne-Mumford
moduli spaces

and the topological vertex is a
way to combinatorially organize our
computation.

Goal: Find a vertex formalism for
$X$ equipped with torus actions

$T \times X \to X$

which have fixed curves.

Examples: (a) local curves (Bryan, Pandharipande)

Take

$\Sigma$ smooth curve of genus $g = g(\Sigma)$

$X = \text{tot} \left( L_1 \oplus L_2 \right)$

$L_i \in \text{Pic} \left( \Sigma \right)$, $\deg L_1 + \deg L_2 = g - 2$

$C^*$ acts linearly along the fibers
fixing the zero section and everything
localizes on this section.
The end computation turns out to be very difficult. Brian Landhuis, ponde solved it by developing a TQFT formalism, degenerate the curve to a curve sewed out of

(b) Local Ruled Surfaces

\[ S = P(L \oplus L) \quad X = \text{tot}(K_S) \]

\[ \lambda \sum_{i} \beta_i^2 (S) \]

We will require

\[ \lambda k + \lambda \varphi = 0. \]
Observation: \( K^0 \) is not ample.

If \( C \neq \emptyset \), then

\[ NS/X|_C = K^0|_C \]

So the moduli of stable maps is not compact.

To repair things we will have to look at the so-called "residual CW theory" (in the terminology of Brian and Pandharipande).

Define

\[ \chi_{hp} = \sum_{t \geq 0} \frac{f(c) c^t}{T(N^{vir})} \]

Note: This definition depends on the choice of a torus action.

- The values of \( \chi_{hp} \) are in general in the character ring of \( T \).
- In certain cases, e.g., in case (b) one can interpret these...
as rational numbers.

Again, the strategy will be to reduce to the case of

\[
\begin{array}{cccc}
\otimes & \otimes & \otimes \\
\end{array}
\]

and then compute for these simple curves.

This will be done by string theory.

We will break $S$ into building blocks

\begin{align*}
R' & \quad R' \\
\text{preimage of} & \quad R \in S \\
\end{align*}

$R, R'$ - Young diagrams

\begin{align*}
(\text{ii}) & \\
R' & \quad R' \\
\text{preimage of} & \quad R \in S \\
\end{align*}
To each of these blocks we will assign a local partition function

\[ V_{\mathcal{R}^1}(g; s, q) \quad \text{for (i)} \]

\[ A_{\mathcal{R}^1}(g; s, q) \quad \text{for (ii)} \]

\[ C_{\mathcal{R}^1}(g; s, q) \quad \text{for (iii)} \]

\[ m_a = \text{self-intersections of components of a reducible fiber of } \mathcal{S} \]

If two of the elementary pieces are sewed together we get a local contribution

\[ V_{\mathcal{R}^1} \cdot \mathcal{C}_{\mathcal{R}^1}(g; s, q; e_1, e_2) \quad \text{for } (i) + (u) \]

\[ V_{\mathcal{R}^1} \cdot \mathcal{C}_{\mathcal{R}^1}(g; s, q; e_1, e_2) \quad \text{for } (i) + (iii) \]

\[ \text{etc.} \]
The formulas for \( V, A, C \) are gotten from strong theory (see slides).