What physicists would like to know about the superpotential

Michael R. Douglas
Rutgers and I.H.E.S.

WAGP05, UPenn

Abstract

Part II.
1. Recap

“Real World” string theory usually works with “$d = 4, N = 1$ effective field theories” (EFTs). These are specified by the following data:

- A gauge group $G$ (with semisimple and finite components).
- A complex Kähler $G$-manifold $C$. In other words, $G$ acts by holomorphic isometries, with real moment maps $\mu_a$.
- A $G$-invariant holomorphic function (in global susy) or section (in supergravity) $W$ on $C$, the superpotential.
- A $G$-invariant holomorphic function $f$ on $C$ (gauge coupling) for each factor in $G$.

This can be thought of as a type of “enhanced moduli space” in which $G$ is introduced to regularize quotient singularities and $W$ parameterizes the obstruction theory. Thus in physical examples, the question naturally arises, what type of objects it parameterizes.
2. **Generalized Mirror Question**

In the cases we discussed last time, we know the answer – for example, Calabi-Yau manifolds, or coherent sheaves on these.

But many, many physics problems start from such concrete or “classical” objects, and then proceed by computing “quantum corrections” to $W$ (and sometimes to the other data). In the best understood cases, these corrections arise from instantons of various types. We then have our question.

In a few of these cases, we know the answer. The most famous starts from large limits of Calabi-Yaus or equivalently $T^n$ fibrations. Adding world-sheet instanton effects to the prepotential, these turn into topological field theories which are mirror to concrete “classical” objects (the B theories). Similarly, one can put certain affine special Lagrangian branes on these limiting CY’s, and then disk instantons turn these into topological boundary conditions, which are again mirror equivalent to more standard objects constructed from coherent sheaves.
But there are many more such problems. Time prohibits giving a detailed survey, but let us list a few with references.

- World-sheet instantons coupling to bundles. These arise in (0,2) models – see Witten 9907041, 0504078; Katz and Sharpe 0406226, Adams et al 0309226.

- Open membranes (Moore, Peradze, Saulina 0012104).

- Divisors on fourfolds (Witten 9604030, Donagi Grassi Witten 9607091, Bergshoeff et al 0507069, Denef et al 0503124).

- Gauge theory instantons (many, many works; see Nekrasov in particular)

and many more.

In many of these problems instantons deform the $W$ arising from some “classical” objects (say, holomorphic bundles) to a “quantum $W$” with a modified obstruction theory, often with fewer moduli. This is of great interest physically (see Eguchi’s talk).

We have very little concrete handle on what are the deformed objects described by these moduli spaces. In the (0,2) case they are (0,2) conformal field theories and this may be the best case to start with. Much of their structure is captured by a chiral algebra of the type defined by Schechtman and collaborators (see
Witten 0504078).

The “generalized mirror hypothesis” would state that in all cases, we can find some classical objects whose moduli space and obstruction theory is being described by these quantum $W$'s. Of course we have no evidence for this in this generality and this is perhaps better stated as the “g.m. question” — can we find conditions on instanton numbers which must hold for this to be true, so we could find evidence for or against the hypothesis?
3. **Dijkgraaf-Vafa theory**

The case of gauge theory instantons is central in physics and has seen a lot of progress. The DV work (2002) was an important development. I will outline a bit of this along the lines of my work with Cachazo, Seiberg and Witten.

We start with a quiver theory, in which $C$ has the standard metric on $C^k$, and the $G$-action defines a linear $G$ representation, call this $R$. Thus the data is $(G, R, W, \mu)$.

The classical moduli space is

$$\frac{R//G}{\{W' = 0\}}.$$
There are various meanings of quantizing this problem. The physical ones are defined as functional integrals from a base $\sigma$ into the moduli space $\mathcal{M}$ (actually, the degrees of freedom of the EFT). Thus they are characterized by the dimension $d$ of the base.

The original case of quantum mechanics is $d = 1$. This leads to $H^*(\mathcal{M})$, BPS algebras, etc.

Conformal field theory (and related massive theories) are $d = 2$ and are like a deformation of $d = 1$.

But the situation changes for $d \geq 3$. In this case one does not get $H^*(\mathcal{M})$ but instead a modified EFT, the “low energy effective field theory” with quantum corrections. Thus one has a map

$$\text{EFT} \rightarrow \text{low energy EFT}$$

In principle this is the result of integrating the renormalization group transformation: given an EFT defined at scale $\Lambda$, we have

$$\text{EFT}(\Lambda) \rightarrow \text{EFT}(\Lambda') \quad \text{with} \quad \Lambda > \Lambda'.$$

In $d = 4$ this only makes sense for certain EFT’s satisfying anomaly cancellation. Given the representation $R$, we consider
the cubic product
\[ R \otimes R \otimes R \]
which defines a vector space of \( G \)-invariants. We then evaluate this on each invariant in \( Sym^3 Lie(G) \). For anomaly cancellation, this must vanish.
In this situation, we can compute a quantum corrected superpotential. There are further rules (which have not been phrased concisely in the physics literature) for when the correction is non-trivial. When it is, the beautiful claim of Dijkgraaf and Vafa is that the correction can be determined in terms of the “zero dimensional quantization” or matrix model

$$Z[W] = \lim_{N \to \infty} \int_{R \otimes M_N} e^{-W}.$$ 

The basic example is $R = \text{Mat}_n(\mathbb{C})$ and

$$W = \sum_{i=1}^{p} t_i \text{Tr} \phi^i.$$ 

Doing the integral produces a family of functions (to be described later) $Z[t; S]$, of $t$ and auxiliary parameters $S$. The quantum corrected $W$ is then a sort of Laplace transform of $S$. 

\[ Z[W] = \lim_{N \to \infty} \int_{R \otimes M_N} e^{-W}. \]

This integral is not as formal as it looks because it is actually to be interpreted as a large \( N \) integral, generating planar diagrams. These always have finite radius of convergence in \( t \), and indeed expectation values can be determined in terms of a simple algebraic system of equations, the Migdal-Makeenko equations.

For each closed loop \( X \) in the quiver, and link \( \phi_i \), we have

\[ \langle \text{Tr} X \frac{\partial W}{\partial \phi_i} \rangle = \sum_{X=X_1 \phi_i X_2} \langle \text{Tr} X_1 \rangle \langle \text{Tr} X_2 \rangle. \]

The right hand side is a sum over all appearances of the link \( \phi_i \) in the loop \( X \).

In the classical problem, the left hand side would be zero. In a sense, the quantization sets it equal to a lower order term.
To make this explicit for the one matrix model, we introduce the generating function (resolvent)

$$ R(z) = \langle \text{Tr} \frac{1}{z - \phi} \rangle. $$

The Migdal-Makeenko equations then become

$$ R(z)W'(z) - f(z) = R(z)^2 $$

where $f(z)$ is the polynomial part of $R(z)W'(z)$. This is easy to solve and this gives the general parameterized solution for $R(z)$; the parameters in $f$ become the parameters $S$.

Finally, writing

$$ W = \sum_{i=1}^{p} t_i \text{Tr} \phi^i, $$

one has

$$ R(z) = \langle \text{Tr} \frac{1}{z - \phi} \rangle = \sum_{i \geq 0} z^{-1-i} \frac{\partial}{\partial t_k} \log Z $$

so one can integrate to get $Z$.

Note that in these terms, the Migdal-Makeenko equations are just the (genus zero) Virasoro constraints on the partition function $Z$. 
We have suppressed some details, for example one really has additional fermions $W_\alpha (\alpha = 1, 2)$ on each node of the quiver, and these must be kept in expectation values as well. One can then make a simple physical proof of these formulas (Cachazo et al) using the “generalized Konishi anomaly,” a four-dimensional version of the invariance under reparameterization

$$\phi \rightarrow \phi + \sum_{n \geq -1} \epsilon_n \phi^{n+1}$$

which led to the Virasoro constraints. Thus the appearance of zero dimensional matrix models in a four dimensional problem is not mysterious.
The equations for quivers with several links

\[ \langle \text{Tr} \; X \frac{\partial W}{\partial \phi_i} \rangle = \sum_{X=X_1 \phi_i X_2} \langle \text{Tr} \; X_1 \rangle \langle \text{Tr} \; X_2 \rangle. \]

are in general quite complicated. Nevertheless they have been solved for a large class of theories (work of Katz, Vafa, and collaborators), those corresponding to branes wrapping cycles in a two-dimensional CY, fibered over a one dimensional base (possibly with monodromy).

In all these cases, one finds that the results can be understood as periods on a new CY\(_3\) \(M'\) obtained by making a geometric transition, along the lines suggested for the conifold by Gopakumar-Vafa. The resulting quantum superpotential is simply a period of the holomorphic three-form,

\[ W_{qu} = \int_{\Sigma} \Omega \]

for a cycle \(\Sigma\) determined by the original ranks of the gauge group, and bare gauge couplings. The moduli of \(M'\) correspond to couplings \(t\) and thus the equations \(DW = 0\) (the attractor mechanism) effectively stabilizes these couplings.
Thus, the generalized mirror hypothesis, in this example, becomes the claim that, starting from any theory obtained from branes wrapping cycles on a CY$_3$, and applying DV, one can express the solution in terms of a new CY$_3$, presumably related by a geometric transition.

In the examples studied so far, this is true so far as one can tell – the equations are generally not solvable, see Berenstein 0303033 and Ferrari 0309151 for the best results so far – with a very interesting exception: there are models in which the relevant geometric transition would be obstructed. For example, branes wrapped on dP$_1$.

In this case, one can argue (Berenstein et al 0505029, Hanany et al, several other groups) that the gauge theory breaks supersymmetry – there is no solution of $DW = 0$. 


It would be important to develop better techniques for understanding the solution to the MM equations

\[
\langle \text{Tr } X \frac{\partial W}{\partial \phi_i} \rangle = \sum_{X=X_1 \phi_i X_2} \langle \text{Tr } X_1 \rangle \langle \text{Tr } X_2 \rangle.
\]

or the corresponding generalized Virasoro constraints. In cases which are deformations of CY categories there might be a nice theory.

The resulting quantum \( W \) defines a moduli space of solutions \( DW = 0 \). What objects does it parameterize?