Deformations of generalized complex structures

Topological $\sigma$-model: this is a QFT describing maps

\[ \mathcal{O} \rightarrow \mathcal{X} \]

The top $\sigma$-model is special since it has an extra operator $\mathcal{Q}: \mathcal{Q}^2 = 0$

The observables are the elements in the $\mathcal{Q}$-cohomology. The metric is needed to define the model but at the end the observables are independent of this metric.

Witten: For a Kähler target there is always a (dilemma) $\sigma$-model. A procedure called topological twisting produces two new theories:

- $A$-model
- $B$-model

They exist always classically but to make sense of them quantum mechanically we need an extra anomaly cancellation.
For the B-model to exist we need \( X \) to be a CY.

If \( X \) is a CY then the best operator for the B-model is:

\[ Q = f \]

and the observables are \( H^p (\Lambda^q T_X) \).

Consider the special case \( p = q = 1 \).

If we have an observable \( o \in H^1 (T_X) \) then \( o \) can be thought of as a complex deformation of \( X \).

Complex infrarational deform can be obstructed i.e., for \( o \) to integrate to an actual deformation we need to have

\[ \text{obs} (o) = 0 \]

where

\[ \text{obs} : H^1 (T_X) \rightarrow H^2 (T_X) \]

is the quadratic obstruction map of Kodaira-Spencer.
However, the theorem of Tran-Todorov says that for a CY

\[ \mathcal{O}_S \cong 0 \]

\[ \Rightarrow \text{every observable in } H^1(T_X) \text{ is an obstruction.} \]

B. G. Kosters extended this to arbitrary \((p,q)\) classes and showed that there exists a smooth extended moduli space parameterizing B-models on a CY \(X\).

It turns out that one can define general topological B-models associated with a generalized CY structure.

Recall: A generalized complex structure is

\[ J : \mathcal{T} \otimes \mathcal{T}^* \rightarrow \mathcal{T} \otimes \mathcal{T}^* \]

with \( J^2 = -\text{id} \) and integrability.

We also have a generalized Kahler structure.
A pair \( (X, J) \) of \( \mathfrak{gc} \) structures is a generalized \( \text{kähler} \) structure if

1. \( IJ = J^*I = -1 \)
2. \(-IJ\) is positive definite.

Roček and co-Authors showed that one can always define a \( (2,1) \) \( \sigma \)-model starting with any generalized \( \text{kähler} \) manifold.

Now we again can construct a \( TQFT \) - a topological twist of the \( (2,1) \) \( \sigma \)-model on \((X, J, J)\).

Again we have two twists corresponding to \( X \) and \( J \) respectively.

(There is no obvious way to order these twists.)

Again we need an anomaly cancellation which turns out to be the generalized \( CY \) condition.
Recall: If $J$ is a GC structure on $M$, then one may find a canonical line bundle $L$ such that $K = A^*(M) \otimes L$ is the annihilator of $\mathfrak{g} = \ker (J - i)$.

Now $(M, J)$ is a generalized CY if $\mathfrak{g} \subset \mathfrak{g} \otimes (M, K)$ satisfies:

- $\mathfrak{g}$ is nowhere zero
- $d\mathfrak{g} = 0$

Now $(M, J)$ is a generalized CY and we can find a complementing $\mathcal{F}$ to $\mathfrak{g}$ into a generalized Kähler structure, then the $(1,1)$-model admits a quantum topological twist corresponding to $J$.

Moreover, $Q \subset \mathfrak{g}$, and the observables are in $H^*(d\mathfrak{g})$.

What about deformations?
Suppose \( p = 2 \) then \( 0 \in H^2(\Omega) \)
corresponds to an infinitesimal defo of \( J \) as a gc structure

It turns out that for a generalized CY manifold, the natural quadratic obstruction map

\[
\text{obs} : H^2(\Omega) \to H^3(\Omega)
\]

always vanishes and that we have an ordinary and extended moduli space of generalized CY structure.

This done by giving a gc version of the Tian-Todorov lemma and the BV deformation formula.

Actually it turns out that the extended moduli space is also a Frobenius manifold.

To see the trace we look at

\[
\mathcal{N}_E \overset{\cdot}{\times} \mathcal{N}_E \quad \text{and we expect to have}
\]

\[
\mathcal{S} : \mathcal{N}_E \to \mathbb{C}
\]
A naive definition would be to take

\[ S(d) = \int \mathbb{R} \times (d+1) \mathbb{R} \]

This is the direct analogue of the complex case.

This does not work and does not give a Frobenius structure. The correct formula is suggested by the interpretation of \( S \) as the one point function in the TQFT.

The answer is gotten as follows. Consider

\[ \tau : T \otimes T^* \]

given by \( \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Now, \( \tau \tau^{-1} \) turns out to be a new gc structure and if we take \( \tilde{S} \) to be the corresponding pure spinor then

\[ \tilde{S}(\tau) = \int \mathbb{R} \times (d+1) \mathbb{R} \]

turns out to be the correct definition.
The 3 tensor of the Frobenius structure is

\[ C_{abc} = \frac{1}{2} \left( a_b \cdot a_c \right) \]

\[ S = \frac{1}{2} \int \epsilon_{\phi \wedge A} \wedge \frac{1}{6} \rho \text{e}^{\phi} e v e n e \]

\( E \) is solution of MC equation.